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Some new results on a linear equation of the second order

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ABSTRACT

Work on solving the second order linear oscillation defined by Eq. (1.1), with continuous and positive coefficient $\Phi(x)$ that satisfies Lipschitz's condition on semi-axis $[0, +\infty)$ and the divergence of $\int_0^{+\infty}(\Phi - G'^2)\,dx$, had started since the 1830s with Sturm's theorems. This paper presents generalizations as well as a simplification of classical Sturm's theorems on the location and the position of zero oscillations, which have not been included in Amrein et al. (2005) [5]. Besides, according to results from Dimitrovski and Mijatović (1997) [1], Dimitrovski et al. (2007) [2] and Dimitrovski et al. (2007) [4], we add some ideas and supplements (Theorems 2.4–2.8, 2.10 and 2.11) to the classical Sturm's theory of oscillations.

1. Introduction and preliminaries

The canonical form of a linear homogeneous differential equation of the second order is

$$y'' + \Phi(x)y = 0 \tag{1.1}$$

under conditions:

1. $\Phi(x)$ satisfies Lipschitz's condition and is a continuous function in area $D = [0, +\infty)$;
2. $\Phi(x) > 0$;
3. $\int_0^{+\infty} \Phi(x)dx$ diverges.

In monographs [1–3], we have presented a new approach to the issues of this equation, using a method we named **series-iteration method** and which not only enables the improvement of Sturm's theorems, but also opens up perspectives for solving new problems.

According to the literature [1,2,4,5], the need for the introduction of the new special function which we named Sturm's functions is established. These will be continuous oscillatory functions with variable amplitudes, periods (that is, distances between zeros), frequencies and pseudo-phases. For the purpose of illustration, the need for their comparison with elementary functions

$$\begin{aligned} y_1 &= F(x)\cos G(x) \\ y_2 &= F(x)\sin G(x) \end{aligned} \tag{1.2}$$

arises, which according to the theorem on the oscillation of the solution to equation (1.1), in an elementary way approximately include the majority of solutions of Eq. (1.1), as well as the connection of coefficients $\Phi(x)$ with amplitude

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$F(x)$ and frequency function $G(x)$, which are approximations of non-elementary solutions of Eq. (1.1), and correspond to Newton's law that is the essence of a complete linear homogeneous differential equation of the second order $y''+ay'+by=0$, and which describe a certain closed system with internal drive and resistances.

A series-iteration method is a classical method of successive approximations of Picard–Lindelöf, but applied on equations with general coefficients, which are continuous and also satisfy Lipschitz's conditions on $[0,+\infty)$. For example, $y''=a(x)y$, $y''=a(x)y'+b(x)y$, $y''=a(x)y^3$. That method has been thoroughly presented in books [1,2], and in journal [4]. It is interesting that it has not been registered in a new monograph. The method consists of orders, whose elements are successive integrals of coefficients of differential equations, where the number of integrals regularly increases. If coefficients are analytical, and for given equation is applicable Cauchy's theorem, a series-iteration method can be derived to power series. Correctness of the method implies from the basic Picard's theorem on existence and uniqueness of solution of differential equation, compared with Cauchy's theorem on analytical solution of analytical differential equation and uniqueness of MacLaurin's series. For the time being, has been determined that the best application of the series-iteration method for ordinary differential equations is on binomial equations in normal form, i.e. with one coefficient, and how the equation of oscillations exactly is (1.1). The major advantage of the method here is generality of coefficients. Surely that expansion on compound equations is possible, but this needs to be developed.

Now we will introduce **Sturm's functions**. If for Eq. (1.1) in normal form we give integral form

$$y(x)=c_1x+c_2-\iint \Phi(x)y\,\mathrm{d}x^2 \tag{1.3}$$

and by selecting the constants $(c_1,c_2)=(0,1)$ and then $(1,0)$, that is, by standardizing with these initial conditions, we have from (1.3) two linearly independent particular integrals y_1 and y_2, which is known from the general theory. For them we define series iteration

$$y_1^{[n]}=1-\iint \Phi(x)y_1^{[n-1]}(x)\mathrm{d}x^2;\qquad y_1^{[0]}=1 \tag{1.4}$$

and

$$y_2^{[n]}=x-\iint \Phi(x)y_2^{[n-1]}(x)\mathrm{d}x^2;\qquad y_2^{[0]}=0 \tag{1.5}$$

where all integrals in iterations should be taken as indefinite and with no limits, because the integrating constants have already been determined by initial conditions.

So, from (1.4) by the iteration technique we obtain solution of Eq. (1.1)

$$y_1=1-\iint \Phi(x)\mathrm{d}x^2+\iint \Phi(x)\mathrm{d}x^2\iint \Phi(x)\mathrm{d}x^2-\iint \Phi(x)\mathrm{d}x^2\iint \Phi(x)\mathrm{d}x^2\iint \Phi(x)\mathrm{d}x^2+\cdots. \tag{1.6}$$

For $\Phi(x)$ positive constant k^2 we have for (1.1): $y''+k^2y=0$, and that is equation of harmonic oscillations. Then from (1.6) implies ordinary Euclidean analytic cosine

$$y_1=1-\frac{(kx)^2}{2!}+\frac{(kx)^4}{4!}-\frac{(kx)^6}{6!}+\cdots=\cos kx.$$

Also, if from (1.5) we develop iterations, we obtain sums of integral

$$y_2=x-\iint x\Phi(x)\mathrm{d}x^2+\iint \Phi(x)\mathrm{d}x^2\iint x\Phi(x)\mathrm{d}x^2-\iint \Phi(x)\mathrm{d}x^2\iint \Phi(x)\mathrm{d}x^2\iint x\Phi(x)\mathrm{d}x^2+\cdots \tag{1.7}$$

where also for $\Phi(x)=k^2$ have oscillations

$$\begin{aligned} y_2&=x-\frac{k^2x^3}{3!}+\frac{k^4x^5}{5!}-\frac{k^6x^7}{7!}+\cdots\\ &=\frac{1}{k}\left[\frac{(kx)^1}{1!}-\frac{(kx)^3}{3!}+\frac{(kx)^5}{5!}-\cdots\right]=\frac{1}{k}\sin kx\end{aligned}$$

and this is also ordinary Euclidean analytic sine of kx.

For series iterations (1.6) and (1.7) numerous characteristics of trigonometric functions $\sin x$ and $\cos x$ can be proven. This gives us the right to introduce the following

Definition 1.1. Functions (1.6) and (1.7) determined by series iterations we call general Sturm's sine and cosine with base $\Phi(x)>0$. We shall mark them with

$$y_1=\cos_{\Phi(x)}x,\qquad y_2=\sin_{\Phi(x)}x. \tag{1.8}$$

For $\Phi(x)$ positive constant we obtain standard Euclidean sine and cosine.

Theorem on oscillatory solution of Eq. (1.1) is known for a long time, where conditions (1–3) are demanded, that is, where $\Phi(x)$ is positive, and large enough to induce oscillations (according to Newton to induce reaction). It also appears that $\Phi(x)$ essentially influences on their characteristics of oscillations, like are zeros, amplitudes, frequencies, inflections, slope, phases; as well as the certain part of formulae of Euclidean Trigonometry.

Definition 1.2. Functions y_1 and y_2 given with (1.6) and (1.7) form generalized Trigonometry of the second order

$$T_{g_2}(\Phi(x)) = T_{g_2}(y_1, y_2) \tag{1.9}$$

which simplifies Euclidean Trigonometry. For $\Phi(x) = k^2 = const.$, $\cos kx$ and $\frac{1}{k}\sin kx$ are already known.

In [2] are also proven **simplification of Sturm's theorems on zero locations of solution of** Eq. (1.1) i.e. on zero of functions (1.6) and (1.7). We are pointing out that theorems on zero locations of solution of differential equation (1.1), and others, are found 100 years after the Sturm's theorems, mainly according to group theories, and for this are credited the following schools: Boruvka, Suyama, Kondratjev. We have proven simple theorems of Sturm's type on location of zeros of equations of oscillations in completely elementary way.

Theorem 1.3. *Zeros of sine solution of Eq. (1.1) are approximately in solutions of equations*

$$x\sqrt{\Phi(x)} = k\pi; \quad k = 0, 1, 2, \ldots \tag{1.10}$$

on semi-axis $[0, +\infty)$*, and at the same points approximately are extreme abscissas of the second cosine solution.*

Theorem 1.4. *Zeros of cosine solution of Eq. (1.1) are approximately in solutions of equations*

$$x\sqrt{\Phi(x)} = (2k-1)\frac{\pi}{2}; \quad k = 1, 2, 3, \ldots \tag{1.11}$$

and at the same points approximately are extreme abscissas of sine solution.

Theorem 1.5. *Inflection points of solutions* y_1, y_2 *are in zeros of those solutions* $y_{1,2}$.

These theorems have great importance:

- geometrical: curves $K = \frac{y''}{(1+y'^2)^{\frac{3}{2}}}$ are in zeros $y_{1,2} = 0$.
- energetic (physical, technical, practical): zeros $y_{1,2} = 0$ also are abscissas of zeros of kinetic energy of oscillation, which is great importance, for example in Electrical engineering.

  According to this are also obtained the simple interpretation of Sturm's functions (1.6) and (1.7) by ordinary Euclidean sine and cosine of complex function.

Theorem 1.6. *For each* $x > 0$ *approximately are correct*

$$\cos_{\Phi(x)} x \approx \cos(x\sqrt{\Phi(x)}) \tag{1.12}$$

$$\sin_{\Phi(x)} x \approx \frac{\sin(x\sqrt{\Phi(x)})}{\sqrt{\Phi(x)}}. \tag{1.13}$$

Here is also important the following.

Definition 1.7. We shall name function $G(x) = x\sqrt{\Phi(x)}$ the frequency function of oscillations determined by Eq. (1.1).

We are pointing out that these results are not mentioned in monograph [5].

2. Main results

Now we give our main results of this paper.

According to above mentioned, we easily obtain Prodi's theorem, which is very important for oscillations stability (for example, second engine run, connected electric circuits).

Theorem 2.1. *Eq. (1.1) with continuous and positive coefficient* $\Phi(x)$*, if defines oscillations when* $x \to +\infty$*, has one limited solution and one solution that pursues to zero.*

Formulae (1.12) and (1.13) suggests to seek solution of Eq. (1.1) in form (1.2) in elementary way, which is solution with amplitude $A = F(x)$ *and one frequency function* $G(x)$*. If we apply differential equations of the second order, we shall look for* *solutions for A: F_1 and F_2 and two solutions for $G = G(x)$. Therefore, in general case we have theoretic possibilities for y_1 and y_2 out of (1.2), and by their substitution in (1.1), we obtain two relations of differential equations*

$$F'' \cos^2 G - 2F'G' \sin G \cos G - F(G'' \sin G \cos G + G'^2 \cos^2 G - \Phi \cos^2 G) = 0$$
$$F'' \sin^2 G + 2F'G' \sin G \cos G - F(-G'' \sin G \cos G + G'^2 \sin^2 G - \Phi \sin^2 G) = 0. \tag{2.1}$$

We obtained nonlinear system and also transcendental regarding the function $G(x)$. If first Eq. (2.1) is multiplied with $\cos G$, and second with $\sin G$, and then summed up, the

$$F'' + (\Phi - G'^2)F = 0 \tag{2.2}$$

will be obtained, which present second order differential equation of canonical type as well as (1.1), if $\Phi(x)$ and $G(x)$ are considered as known.

Theorem 2.2. *In order to Eq. (1.1) has solution in the form (1.2) where $F(x)$ are amplitudes of oscillatory solutions, $G(x)$ is function of oscillations frequency, $\Phi(x)$ is cause of oscillations (with calculated resistances) and characteristics: $\Phi(x)$ is continuous and satisfies Lipschitz's condition on semi-axis $[0, +\infty)$, $\Phi(x) > 0$ and $\int_0^{+\infty}(\Phi - G'^2)\mathrm{d}x$ diverges.*

Necessary condition for this is that connection (2.2) applied as well as the important definition.

Definition 2.3. We shall name Eq. (2.2) **amplitude equation** for (1.1).

According to given results from [1,2,4], to the classical Sturm's theory of oscillations we could add up the following ideas and supplements:

(A) Two-amplitude and two-periodic solutions; (B) Uniperiodic and two-amplitude solutions and **(C) Parabolic case of equation.**

(A). For given $\Phi(x)$, let in (15) is also applicable $\Phi(x) - G'^2(x) > 0$, and let integral $\int_0^{+\infty}(\Phi - G'^2)\mathrm{d}x$ diverges.

In that case, solutions for amplitudes $F(x)$ are also oscillatory functions, which can be determinate by Sturm's functions (1.8).

Theorem 2.4. *If (2.2) also presents one oscillatory equation of the second order, that is, are applied conditions under* (A)*, then amplitudes F_1 and F_2 are also Sturm's functions*

$$F_1 = \cos_{(\Phi - G'^2)} x, \qquad F_2 = \sin_{(\Phi - G'^2)} x \tag{2.3}$$

and general solution for amplitude is $F(x) = \alpha \cos_{(\Phi - G'^2)} x + \beta \sin_{(\Phi - G'^2)} x$, where (α, β) are new integrating constants.

Then according to [2] are applied approximate formulae for solution by ordinary Euclidean sines and cosines and also for amplitudes.

Theorem 2.5. *For oscillatory solutions (1.2), for each $x > 0$, approximately are correct*

$$A_1 = F_1 = \cos_{(\Phi - G'^2)} x \approx \cos(x\sqrt{\Phi - G'^2}) \tag{2.4}$$

$$A_2 = F_2 = \sin_{(\Phi - G'^2)} x \approx \frac{\sin(x\sqrt{\Phi - G'^2})}{\sqrt{\Phi - G'^2}}. \tag{2.5}$$

That is, oscillations amplitudes are not equal and are oscillatory functions too, and for them applied new Prodi's theorem.

Theorem 2.6. *Under conditions in* (A)*, for Eq. (2.2) amplitudes F_1 and F_2 have asymptotic behaviour: $F_1 = A_1$ remains limited, $F_2 = A_2$ is amplitude that tends zero when $x \to +\infty$.*

And for equation solution (1.1) alone, according to our interpretation of Sturm's functions (1.8), from (1.2) we have $y = c_1F_1 \cos G(x) + c_2F_2 \sin G(x)$.

Here follows the theorem.

Theorem 2.7. *If in (1.1) and (1.2) are applied conditions under* (A)*, then the solutions of Eq. (1.1) are oscillatory and two-amplitude and are*

$$y_1 = \cos_{(\Phi - G'^2)} x \, \cos G(x), \tag{2.6}$$

$$y_2 = \sin_{(\Phi - G'^2)} x \, \sin G(x), \tag{2.7}$$

or in closer form, introducing by (2.4) and (2.5) Euclidean sine and cosine

$$y_1 \approx \cos(x\sqrt{\Phi - G'^2}) \cos G(x) \quad \text{and} \quad y_2 \approx \frac{\sin(x\sqrt{\Phi - G'^2})}{\sqrt{\Phi - G'^2}} \sin G(x). \tag{2.8}$$

Pseudo-periods of solutions are

$$T_1 = \frac{2\pi}{\sqrt{\Phi - G'^2}}, \qquad T_2 = \frac{2\pi x}{G(x)}, \tag{2.9}$$

and are also two-amplitude, and amplitudes are

$$A_1 = 1, \qquad A_2 = \frac{1}{\sqrt{\Phi - G'^2}}. \tag{2.10}$$

Term "pseudo-" (-period, -amplitude, -phase) here considers Sturm's variable periods (amplitudes, phases)-those are distances between two zeros, between which is also the third zero that separates positive and negative part of oscillation. Also, from (2.8) we have Prodi's theorem for asymptotic behaviour for this case, since $\sqrt{\Phi - G'^2} \to \infty$, when $x \to +\infty$.

Theorem 2.8. *Under condition of Theorem 2.7, one solution y_1 remains limited and second solution $y_2 \to 0$, when $x \to +\infty$.*

(B). Let us now study the second possibility of sign in (2.2). Let (2.2) can be written with $\Phi(x) - G'^2(x) < 0$. Then (2.2) can be written with

$$F'' - |\Phi - G'^2|F = 0, \tag{2.11}$$

or in normal form $F'' = |\Phi - G'^2|F$, where coefficient is always positive. From integral form $F = c_1x + c_2 + \iint |\Phi - G'^2|F(x)\mathrm{d}x^2$, using the same method of series iteration [1,2] can be shown that there are two linearly independent particular solutions

$$\begin{aligned} F_1 &= 1 + \iint |\Phi - G'^2|\mathrm{d}x^2 + \iint |\Phi - G'^2|\mathrm{d}x^2 \iint |\Phi - G'^2|\mathrm{d}x^2 \\ &\quad + \iint |\Phi - G'^2|\mathrm{d}x^2 \iint |\Phi - G'^2|\mathrm{d}x^2 \iint |\Phi - G'^2|\mathrm{d}x^2 + \cdots \end{aligned} \tag{2.12}$$

as well as

$$\begin{aligned} F_2 &= x + \iint x|\Phi - G'^2|\mathrm{d}x^2 + \iint |\Phi - G'^2|\mathrm{d}x^2 \iint x|\Phi - G'^2|\mathrm{d}x^2 \\ &\quad + \iint |\Phi - G'^2|\mathrm{d}x^2 \iint |\Phi - G'^2|\mathrm{d}x^2 \iint x|\Phi - G'^2|\mathrm{d}x^2 + \cdots \end{aligned} \tag{2.13}$$

which are seen to be monotonously increasing, and that for $|\Phi - G'^2| = const. = a^2$ give hyperbolic functions $\cosh ax$ and $\frac{1}{a}\sinh ax$. Therefore we can introduce new class of Sturm's functions.

Definition 2.9. Solutions of Eq. (2.11) for amplitudes are called Sturm's hyperbolic functions with base $|\Phi - G'^2|$, and are marked with $F_1 = \cosh_{|\Phi - G'^2|} x$, $F_2 = \sinh_{|\Phi - G'^2|} x$, and they are also given by series iterations

$$A_1 = F_1 = \frac{1}{2}(\mathrm{e}^{x\sqrt{G'^2 - \Phi}} + \mathrm{e}^{-x\sqrt{G'^2 - \Phi}}), \tag{2.14}$$

$$A_2 = F_2 = \frac{1}{2}(\mathrm{e}^{x\sqrt{G'^2 - \Phi}} - \mathrm{e}^{-x\sqrt{G'^2 - \Phi}}). \tag{2.15}$$

Generalized hyperbolic functions (2.12) and (2.13) are defined in usual way and majority relations of real hyperbolic trigonometry apply here.

General solution (2.12) for $F(x)$ is $F(x) = \alpha \cosh_{|\Phi - G'^2|} x + \beta \sinh_{|\Phi - G'^2|} x$, and from hyperbolic trigonometry we know that $F(x)$ can have at most one zero. Then from (2.11), on the base [1] or [2] follows.

Theorem 2.10. *Solutions of Eq. (2.11) are monotonously increasing amplitudes of non-monotonous oscillatory solution of Sturm's type Eq. (1.1), and those are*

$$y_1 = \cosh_{|\Phi - G'^2|} x \, \cos G(x), \tag{2.16}$$

$$y_2 = \sinh_{|\Phi - G'^2|} x \, \sin G(x) \tag{2.17}$$

where pseudo-periods are

$$T = \frac{2\pi x}{G(x)} \tag{2.18}$$

and amplitudes (for (2.14) and (2.15)) are

$$A_1 = \cosh(G'^2 - \Phi), \qquad A_2 = \frac{\sinh(G'^2 - \Phi)}{\sqrt{G'^2 - \Phi}}, \tag{2.19}$$

and they monotonously increase when $x \to +\infty$, keeping the relation $A_2 < A_1$.

From here also applies the following.

Theorem 2.11 (*Sturm's Theorem on Zeros of the Oscillatory Functions*)**.** *Solution of Eq. (2.11) have the most one zero in each interval inside semi-axis $x \geq 0$.*

(C). If in (1.1) is special

$$\Phi(x) = G'^2(x) \tag{2.20}$$

then Eq. (2.12) is $F'' = 0$. Its solution is linear function $F(x) = ax+b$. As is from (2.18) frequency function $G(x) = \int \sqrt{\Phi(x)}\mathrm{d}x$, solutions of equation of oscillations (1.1) are

$$y_1 = (ax+b)\cos\int\sqrt{\Phi(x)}\mathrm{d}x \quad \text{and} \quad y_2 = (ax+b)\sin\int\sqrt{\Phi(x)}\mathrm{d}x \tag{2.21}$$

with Sturm's zeros (for y_1 in solutions of equation $\int\sqrt{\Phi(x)}\mathrm{d}x = (2k-1)\frac{\pi}{2};\ k = 1, 2, 3, \ldots$, and for y_2 in solutions of equation $\int\sqrt{\Phi(x)}\mathrm{d}x = k\pi;\ k = 0, 1, 2, \ldots$) and with eventual zero $ax+b=0$, if positive.

3. Conclusion

By this, analysis of oscillatory equation (1.1) is exhausted, in the form of solution (1.2), in the case of continuous coefficient $\Phi(x)$. We can see that diversity is not particularly big: those are either variable oscillations, but with great regularity in disposition of zeros, "periods", frequencies, amplitudes; or those are monotonous hyperbolic functions. Real and great diversity is to be expected when in oscillation (1.1) $\Phi(x)$ is discontinuous coefficient.

The following examples illustrate applicability of our main result on existence of oscillatory solutions and exact locations of zeros.

Example 3.1. We shall form a differential equation (1.1) if for particular integrals we select $y_1 = \sin(x^{1-\frac{\alpha}{2}})$ and $y_2 = \cos(x^{1-\frac{\alpha}{2}})$. These integrals are oscillatory if $x \to +\infty$ for $0 < \alpha < 2$.

First it is necessary that Wronskian $W = y_1'y_2 - y_1y_2'$ is different from zero. After elementary calculation, from $\begin{vmatrix} y'' & y' & y \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix} = 0$, we obtain linear homogeneous differential equation

$$y'' + \frac{\alpha}{2}\frac{1}{x}y' + \left(1 - \frac{\alpha}{2}\right)^2 x^{-\alpha}y = 0. \tag{3.1}$$

Using the well-known Liouville formula $y = \exp(-\frac{1}{2}\int\Phi(x)\mathrm{d}x)z = \frac{1}{\sqrt[4]{x^\alpha}}z$ we obtain the canonical form of Eq. (3.1)

$$z'' + \Phi(x)z = 0$$

where $\Phi(x) = (1-\frac{\alpha}{2})^2x^{-\alpha} + (\frac{\alpha}{4} - \frac{\alpha^2}{16})\frac{1}{x^2}$.

It follows

(**1**) For $\alpha = 0$ there is $\Phi(x) = 1$ and the equation transforms to harmonic oscillation equation $z'' + z = 0$.

(**2**) For $\alpha = 1$ there is $\Phi(x) = \frac{1}{4x} + \frac{3}{16x^2}$ and hence $z'' + (\frac{1}{4x} + \frac{3}{16x^2})z = 0$. According to Sansone [6], this equation has general solution $z = c_1\sqrt[4]{x}\cos\sqrt{x} + c_2\sqrt[4]{x}\sin\sqrt{x}$ which is oscillatory because $\Phi(x)$ is approximated with $\frac{1}{4x}$. This absolutely complies with the classical theory of oscillations because integral $\int_0^{+\infty}\Phi(x)\mathrm{d}x$ diverges.

Solutions of Eq. (3.1) could be $z_1 = \sin_{\Phi(x)} x$, $z_2 = \cos_{\Phi(x)} x$ or $z_1 = \cosh_{\Phi(x)} x$, $z_2 = \sinh_{\Phi(x)} x$. If we choose $\Phi(x) \sim 0$, then as solution we get linear function $z = c_1x + c_2$.

The zeros of solution are in the solutions of following equations

for the sine solution $x\sqrt{\Phi(x)} = x\sqrt{(1-\frac{\alpha}{2})^2x^{-\alpha} + (\frac{\alpha}{4} - \frac{\alpha^2}{16})\frac{1}{x^2}} = n\pi,\ n = 0, 1, 2, \ldots$

for the cosine solution $x\sqrt{\Phi(x)} = x\sqrt{(1-\frac{\alpha}{2})^2x^{-\alpha} + (\frac{\alpha}{4} - \frac{\alpha^2}{16})\frac{1}{x^2}} = (2n-1)\frac{\pi}{2},\ n = 1, 2, \ldots.$

The amplitudes are $A_1 = \frac{1}{\sqrt[4]{x^\alpha}}$, $A_2 = \frac{1}{\sqrt[4]{x^\alpha}}\frac{1}{\sqrt{\Phi(x)}}$.

Now we shall give two examples for which the classical theorem of oscillation is not applicable.

Example 3.2. Let us consider differential equation $y'' + \frac{1}{(1+x)^{\frac{3}{2}}} y = 0$, where $\Phi(x) = (1+x)^{-\frac{3}{2}}$ is approximated with $\frac{1}{x^{\alpha}}$. Since $0 < \alpha = \frac{3}{2} < 2$, according to our theorem the solutions $y_1 = \sin_{\Phi(x)} x$, $y_2 = \cos_{\Phi(x)} x$ are oscillatory. Since the arguments of sine and cosine in the form $G(x) = x\sqrt{\Phi(x)} = x(1+x)^{-\frac{3}{4}} \sim x^{\frac{1}{4}}$ increase fast enough, it implies that there are cross sections of $G(x)$ with horizontals $y = n\pi$. Hence, the solution is oscillatory.

However, according to the classical theory, integral $\int_0^{+\infty} \Phi(x)\mathrm{d}x = 2$ converges so the oscillations are not possible to define.

Example 3.3. The same also applies if $\Phi(x) = \frac{1}{1+x^2} \sim \frac{1}{x^2}$. From $\int_0^{+\infty} \Phi(x)\mathrm{d}x = \frac{\pi}{2}$ we do not have a solution to oscillations. However, as in the previous example, if we use the frequency function $G(x) = x\sqrt{\Phi(x)} = \frac{x}{\sqrt{1+x^2}}$ we get that equation $\frac{x}{\sqrt{1+x^2}} = n\pi$, $n = 0, 1, 2, \ldots$ has a solution, which does not have zeros except $x = 0$ for $n = 0$. In other cases, there are no cross sections.

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